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Noncommutative geometry and gauge theory on discrete groups

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Abstract

We build and investigate a pure gauge theory on arbitrary discrete groups. A systematic approach to the construction of the differential calculus is presented. We study the metric properties of the models and introduce the action functionals for unitary gauge theories. A detailed analysis of two simple models based on $\mathbb{Z}_2$ and $\mathbb{Z}_3$ follows. Finally we discuss briefly the models with additional symmetries.

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1. Introduction and notation

The noncommutative geometry provides us with a far more general framework for physical theories than the usual approaches. Its basic idea is to substitute an abstract, associative and not necessarily commutative algebra for the algebra of functions on a smooth manifold [1-5]. This allows us to use nontrivial algebras as a geometrical setup for field-theoretical purposes, in particular for gauge theories, which are of special interest both from mathematical and physical point of view.

The construction of noncommutative gauge theories has led to a remarkable result, which is the description of the Higgs field in terms of a gauge potential. This suggests

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some possible nontrivial geometry behind the structure of the Standard Model. Several examples of this kind, with various choices of fundamental objects of the theory, have been investigated in such a context (see [5–13] and references therein). The "discrete geometry" models, which take as the algebra the set of functions on a discrete space, seem to be one of the most promising interpretations [4,5] and suggest that such a geometry may play an important role in physics. Recently, some more analysis has been carried out for discrete spaces [14,15] in the context of grand unification and general relativity.

We propose to develop here a systematic approach towards the construction of field theories, in particular the gauge theory, on arbitrary discrete spaces equipped with the group structure. The choice of a group as our base space allows us to make use of the correspondence with the differential calculus on Lie groups. We define the objects, which correspond to invariant vector fields (though they are not derivations) and forms (dual to the latter), building the differential calculus on the discrete space.

We introduce also the concept of a metric on the differential algebra, which allows us to construct the actions of the fundamental physical models. We demonstrate that, in general, the ambiguity in the possible form of the action is much broader than in the case of continuous theories.

The paper is organised as follows: in the first section we construct the tools of the differential calculus. We present a new way of constructing such a calculus in noncommutative geometry, introducing the concept of vector fields. We develop also a new approach to the metric in noncommutative geometry, introducing the analogue of metric tensor.

Next we outline the general formalism of gauge theories in this case and some problems of the construction of actions, outlining the existence of many possibilities. The discussion of two examples follows. Finally we present other possibilities originating from the symmetry principles, the results of combining the discrete and continuous geometry.

2. Differential calculus

Let G be a finite group and $\mathcal{A}$ be the algebra of complex valued functions on G. We will denote the group multiplication by $\odot$ and the size of the group by N_G. The right and left multiplications on G induce natural automorphisms of $\mathcal{A}$, R_g and L_g, respectively,

$$(R_h f)(g) = f(g \odot h), \tag{1}$$

with a similar definition for L_g.

Now we will construct the extension of $\mathcal{A}$ into a graded differential algebra. We shall attempt to follow the standard procedure of introducing the differential calculus on manifolds, in particular on Lie groups. Therefore we shall use almost the same terminology, though the properties of certain objects may differ from the usual ones.

First let us identify the space of vector fields V over $\mathcal{A}$ with the linear operators on $\mathcal{A}$, which have their kernel equal to the space of constant functions. Let us point out here that we require no other condition, in particular, the Leibniz rule is not obeyed.

Such operators form a finite dimensional left module over $\mathcal{A}$. Now, we can define the vector space $\mathcal{F}$ of left invariant vector fields as satisfying the following identity:

$$\partial \in \mathcal{F} \quad \Leftrightarrow \quad \forall f \in \mathcal{A},\ L_h\partial(f) = \partial(L_h f). \tag{2}$$

Before we discuss the algebraic structure of $\mathcal{F}$ let us observe that this vector space is $N_G - 1$ dimensional and it generates the module of vector fields. Therefore for a given basis of $\mathcal{F}$, $\{\partial_i\}_{i=,1\dots,N_G}$, every vector field can be expressed as a linear combination $f_i\partial_i$, with the coefficients f_i from the algebra $\mathcal{A}$.

$\mathcal{F}$ forms an algebra itself and we find the relations of the generators to be of second order,

$$\partial_i\partial_j = \sum_k C_{ij}^k \partial_k, \tag{3}$$

where C_{ij}^k are the structure constants. Because of the associativity of the algebra they must obey the following set of relations,

$$\sum_l C_{ij}^l C_{lk}^m = \sum_l C_{il}^m C_{jk}^l. \tag{4}$$

Now we choose a specific basis of $\mathcal{F}$ and calculate the relations (3) in this particular case. It is convenient for our purposes to introduce the basis of $\mathcal{F}$ labelled by the elements of $G' = G \setminus \{e\}$, where e is the neutral element of G. Further on, if not stated otherwise, it should be assumed that all indices take values in G'. We define:

$$\partial_g f = f - R_g f, \quad g \in G',\ f \in \mathcal{A}. \tag{5}$$

The structure relations (3) become quite simple in the chosen basis,

$$\partial_g\partial_h = \partial_g + \partial_h - \partial_{(h\odot g)}, \quad g, h \in G'. \tag{6}$$

As a next step let us introduce the Haar integral, which is a complex valued linear functional on $\mathcal{A}$ that remains invariant under the action of both R_g and L_g,

$$\int f = \frac{1}{N_G}\sum_{g\in G} f(g), \tag{7}$$

where we normalised it so that $\int 1 = 1$.

Although the elements of $\mathcal{F}$ do not satisfy the Leibniz rule, they are, in a sense, inverse to the integration. Indeed, we notice that for every $f \in \mathcal{A}$ and every $v \in \mathcal{F}$ the integral (7) of $v(f)$ vanishes. For this reason we can consider them as corresponding to the derivations on the algebra $\mathcal{A}$.

We define now the space of one-forms Ω^1 as a right module over $\mathcal{A}$, which is dual to the space of vector fields. Of course, we can also introduce the notion of left invariant forms, which, when acting on the elements of $\mathcal{F}$, give constant functions.

Having chosen the basic of $\mathcal{F}$ we automatically have the dual basis of $\mathcal{F}^*$ consisting of forms χ^g, $g \in G'$, which satisfy

$$\chi^g(\partial_h) = \delta^g_h. \tag{8}$$

To build a graded differential algebra we need to construct n-forms and their products for an arbitrary positive integer n. Of course, we identify zero-forms with the algebra $\mathcal{A}$ itself and their product with the product in the algebra. The definition for higher forms is natural, we take Ω^n to be the tensor product of n copies of Ω^1,

$$\Omega^n = \underbrace{\Omega^1 \otimes \cdots \otimes \Omega^1}_{n \text{ times}}, \tag{9}$$

and the product of forms to be the tensor product over $\mathcal{A}$. We have assumed here that Ω^1 is a bimodule, with algebra $\mathcal{A}$ acting from both sides, which may not be the same. In fact, it appears that the left action could be defined only when introducing the differential structure on the algebra Ω:

Lemma 1. *There exists exactly one linear operator d, $d : \Omega^n \to \Omega^{n+1}$, which is nilpotent, $d^2 = 0$, satisfies the graded Leibniz rule and for every $f \in \mathcal{A}$ and every vector field v, $df(v) = v(f)$, provided that the right and left action of $\mathcal{A}$ on $\mathcal{F}^*$ are related as follows,*

$$f\chi^g = \chi^g(R_g f), \quad g \in G', \ f \in \mathcal{A}, \tag{10}$$

and that the following structure relations hold,

$$d\chi^g = \sum_{h,k} C^g_{hk} \chi^k \otimes \chi^h, \quad g \in G'. \tag{11}$$

Before we prove the lemma, let us observe that since the invariant one-forms are a basis of Ω^1 the left action of $\mathcal{A}$ (10) could be extended to all one-forms. The next requirement (11) is equivalent to the Maurer–Cartan structure relation on Lie groups.

Proof. Since we want d to satisfy the graded Leibniz rule, it is sufficient to define the action of d on $\mathcal{A}$ and on $\mathcal{F}^*$ because all other forms can be represented as tensor products of them. The action of d on $\mathcal{A}$ is defined by the requirement stated in the lemma, from which we get that

$$df = \sum_g \chi^g(\partial_g f). \tag{12}$$

The Leibniz rule applied to the product of any two elements $a, b \in \mathcal{A}$, gives the following identity:

$$\sum_g \chi^g\left(ab - R_g(a)R_g(b)\right) = \sum_g \chi^g\left(a - R_g(a)\right)b + a\chi^g\left(b - R_g(b)\right), \tag{13}$$

which is satisfied only if (10) holds. The Maurer–Cartan relations arise from the requirement that d^2 acting on an arbitrary $a \in \mathcal{A}$ must vanish. Indeed, we calculate,

$$d^2a = d\left(\sum_h \chi^h(\partial_h a)\right)$$
$$= \sum_{h,k} -\chi^h \otimes \chi^k C^g_{kh}(\partial_g a) + \sum_h d\chi^h(\partial_h a), \tag{14}$$

and this expression vanishes only if (11) is true.

If for any vector field $v = \sum_g v^g \partial_g$ we calculate $da(v)$:

$$da(v) = \sum_{g,h} v^g \chi^h(\partial_g)\partial_h(a) = \ldots$$

and since χ^g is a basis dual to ∂_h we finally obtain:

$$\ldots = \sum_g v^g \partial_g(a) = v(a),$$

so we have verified that the external derivative has the same property as in the usual differential geometry. This ends the proof. □

In our construction we have obtained the differential algebra over the algebra of complex functions on a discrete group, which may be the starting point for the analysis of this structure.

The differential calculus presented here is equivalent to the universal differential calculus and therefore it does not depend on the group structure of the discrete space, which remains then only a convenient tool. We shall demonstrate later that the group structure is important when considering other examples of differential calculi resulting from the one discussed above.

Let us end this section by constructing the involution on our differential algebra, which agrees with the complex conjugation on $\mathcal{A}$ and (graded) commutes with d, i.e. $d(\omega^\star) = (-1)^{\deg \omega}(d\omega)^\star$. Again, it is sufficient to define it for the basis of one-forms,

$$(\chi^g)^\star = -\chi^{g^{-1}}. \tag{15}$$

So far, we restricted ourselves in our approach to the complex-valued functions. Similarly we can consider a straightforward extension of the model if we take functions valued in any involutive algebra, for instance, the matrix valued functions. The quotient subalgebras of the algebra obtained may also be considered, and we shall briefly discuss it in the last section.

3. Gauge theory

3.1. General formalism

In this section we shall construct the gauge theory on finite groups using the differential calculus we have just introduced. First let us explain some basic ideas. The starting point is the differential algebra $\tilde{\Omega}^*$ with its subalgebra of zero-forms $\tilde{\mathcal{A}} \subset \tilde{\Omega}^*$. We take the group of gauge transformations to be any proper group $\mathcal{H} \subset \tilde{\mathcal{A}}$ which generates $\tilde{\mathcal{A}}$. In particular, we shall often take $\mathcal{H}$ to be the group of unitary elements of $\tilde{\mathcal{A}}$,

$$\mathcal{H} = \mathcal{U}(\tilde{\mathcal{A}}) = \{a \in \tilde{\mathcal{A}} : aa^\star = a^\star a = 1\}.$$

Of course, the external derivative d is not covariant with respect to the gauge transformations. Therefore we have to introduce the covariant derivative $d+\Phi$, where Φ is a one-form. The requirement that $d+\Phi$ is gauge covariant under gauge transformations,

$$d+\Phi \;\rightarrow\; H^{-1}(d+\Phi)H, \quad H \in \mathcal{H}, \tag{16}$$

results in the following transformation rule of Φ,

$$\Phi \;\rightarrow\; H^{-1}\Phi H + H^{-1}dH. \tag{17}$$

Φ is the gauge potential, which we will also call connection. If the gauge group is unitary, we require also that the covariant derivative is hermitian,

$$d\left(a^\star b\right) \;=\; a^\star(d+\Phi)b + \left(b^\star(d+\Phi)a\right)^\star, \quad a,b \in \tilde{\mathcal{A}}, \tag{18}$$

which results in the condition that the connection is anti-selfadjoint, $\Phi = -\Phi^\star$. Finally, we have the curvature two-form, $F = d\Phi + \Phi\Phi$, which, of course, is gauge covariant.

3.2. Gauge transformations, connection and curvature on discrete groups

Let us take the algebra $\tilde{\mathcal{A}}$ to be the tensor product of the algebra $\mathcal{A}$ of complex valued functions on G, which we introduced in the previous section, by a certain algebra A, which could be the algebra of complex $n \times n$ matrices M_n, for instance. In such a case, the differential algebra $\tilde{\Omega}^*$ is clearly the tensor product of Ω^* by A. The group of gauge transformations, as defined above, can be identified with the group of functions on G taking values in a group $\mathsf{H} \subset \mathsf{A}$. Similarly, gauge potentials are interpreted as A valued one-forms. We will denote the involution on A by $\dagger$.

Before we introduce the metric and present the Yang–Mills actions, let us work out the gauge transformation rules (17) for the connection and the curvature in the convenient basis we chose (8) in our paper. If we write $\Phi = \sum_g \chi^g \Phi_g$, the transformation of Φ_g under a gauge transformation $H \in \mathcal{H}$ is,

$$\Phi_g \;\rightarrow\; (R_g H)^{-1}\Phi_g H + (R_g H)^{-1}\partial_g H. \tag{19}$$

The condition $\Phi = -\Phi^\star$ enforces the following relation of its coefficients,

$$\Phi_g^\dagger \;=\; R_g\left(\Phi_{g^{-1}}\right). \tag{20}$$

If we introduce a new field $\Psi_g = 1 + \Phi_g$, we can see that (19) is equivalent to

$$\Psi_g \;\rightarrow\; (R_g H)^{-1}\Psi_g H. \tag{21}$$

The introduction of Ψ_g is convenient for the calculations as it simplifies the formulas. We will discuss the physical meaning of this step later. It is instructive to compare the formulas for the coefficients of the curvature,

$$F = \sum_{g,h} \chi^g \otimes \chi^h F_{gh},$$

using both Φ_g and Ψ_g. In the first case, we obtain from the definition of F, the rules of differential calculus (10), (11) and the exact form of the structure constants in this basis (5),

$$F_{gh} = \Phi_h + R_h(\Phi_g) + R_h(\Phi_g)\Phi_h - \Phi_{(g\odot h)}, \tag{22}$$

whereas the same formula written using Ψ_g is much simpler,

$$F_{gh} = \Psi_{(g\odot h)} - R_h(\Psi_g)(\Psi_h). \tag{23}$$

The transformation rule for F_{gh} follows from the gauge covariance of F. However, since the algebra is noncommutative the coefficients are no longer gauge covariant:

$$F_{gh} \to R_{(g\odot h)}(H^{-1})F_{gh}H. \tag{24}$$

In order to proceed with the construction and analysis of the Yang-Mills theory we have to introduce a metric.

3.3. Metric

In this subsection we shall briefly discuss a general concept of a metric tensor in noncommutative geometry [11-13] and we shall concentrate on the analysis of the model with discrete geometry.

Let us define the metric η as a form on the left module of one-forms, valued in the algebra $\tilde{\mathcal{A}}$ and middle-linear over the algebra $\tilde{\mathcal{A}}$,

$$\begin{aligned} &\eta : \tilde{\Omega}^1 \times \tilde{\Omega}^1 \mapsto \tilde{\mathcal{A}}, \\ &\quad \eta(avc, ub) = a\eta(v, cu)b, \quad a, b, c \in \tilde{\mathcal{A}},\ u, v \in \tilde{\Omega}^1. \end{aligned} \tag{25}$$

Note that η can no longer be symmetric; however we may require that it is hermitian:

$$\eta(u,v)^\star = \eta(v^\star, u^\star), \quad \forall u, v \in \tilde{\Omega}^1 \tag{26}$$

Let us observe the properties of this metric on the discrete space. Since the invariant forms χ^g are the basis of $\tilde{\Omega}^1$ the metric is determined by its coefficients $\eta^{gh} = \eta(\chi^g, \chi^h)$. Now, from the property of middle-linearity (25) and the rules of left and right multiplication (10) we obtain

$$a\eta^{gh} = \eta^{gh} R_{g\odot h} a, \quad g, h \in G', a \in \mathcal{A}, \tag{27}$$

which can be satisfied only if η^{gh} vanishes for $g \neq h^{-1}$. Therefore, the metric must have the following form:

$$\eta^{gh} = E_g \delta^{g^{-1}h)}, \tag{28}$$

where E_g are arbitrary elements of $\mathcal{A}$. If we assume that the metric is hermitian, we must have $E_g = E_g^\star$. If we look closely at these conditions (28) we shall see that the metric is in fact diagonal, as for any one form $v = \sum_g \chi^g v_g$ we have

$$\eta(v^\star, v) = \sum_g E_{g^{-1}} v_g^\star v_g.$$

It is rather surprising that in general we have that $\eta(v^\star, v) \neq \eta(v, v^\star)$; however after integrating both sides we may recover the equality only if $E_g = R_{g^{-1}} E_{g^{-1}}$.

Finally let us briefly touch on the subject of extension of the metric to the forms of higher order. Since any two-form can be decomposed into a sum of products of one-forms, let us observe that we may define two different middle-linear functionals on $\tilde{\Omega}^2$:

$$\theta_1(a_1 \otimes b_1, a_2 \otimes b_2) = \eta(a_1, b_1)\eta(a_2, b_2), \tag{29}$$

$$\theta_2(a_1 \otimes b_1, a_2 \otimes b_2) = \eta(a_1, \eta(b_1, a_2) b_2). \tag{30}$$

It is easy to check that they are also hermitian in the sense of definition (26).

Let us point out that the existence of two different middle-linear functionals built from the metric on the module of two-forms is a new property which appears in the noncommutative geometry. In the classical differential geometry one of them is trivial due to the anticommutativity of differential forms.

Another interesting property of the considered differential calculus is the existence of a linear functional $\theta_l : \tilde{\Omega}^2 \to \mathcal{A}$, defined simply as:

$$\theta_l(a \otimes b) = \eta(a, b). \tag{31}$$

It is easy to see that (29) is simply a product of this linear functional.

We shall use all these functionals to construct the actions in models of gauge theory on discrete spaces.

3.4. The Yang–Mills action

We would like the Yang–Mills action to be constructed in the same way as in the case of the gauge theories on manifolds. Therefore, we postulate that for an involutive algebra and the structure group $\mathsf{H} \subset \mathcal{U}(\mathsf{A})$, it has the following form,

$$S_{\mathrm{YM}} = \int_G \mathcal{L}(\Phi), \tag{32}$$

where $\int_G$ is the Haar integral on $\mathcal{A}$ and $\mathcal{L}(\Phi)$ is the gauge invariant Lagrangian belonging to the algebra $\mathcal{A}$, which is of second order in the curvature F. First of all, from our discussion of the metric we notice that there are many possibilities of such actions. We encounter an ambiguity in the choice of the middle-linear functional on the module of two-forms, as any linear combination of θ_1 and θ_2 (29), (30) can be used. Moreover, we have the ambiguity which arises from the difference between $\theta(F^\star, F)$ and $\theta(F, F^\star)$.

Finally, due to the existence of (31) we may admit an additional gauge invariant term linear in the curvature F, which is also a new feature of our theory.

Now, let us calculate explicitly all these actions. We assume only the hermiticity of the metric, having in mind that further constraints shall reduce the number of possibilities.

Using the functional θ_2 (30) we get the usual Yang–Mills lagrangian, and after integrating we have:

$$S_{\rm YM} = \int\limits_G \sum_{g,h} (R_h E_{g^{-1}}) E_{h^{-1}} F^\star_{gh} F_{gh} \,. \tag{33}$$

If we take $\theta_2(F, F^\star)$ instead of $\theta_2(F^\star, F)$ we shall get the same kind of action as above, however with a redefined metric:

$$E_{g^{-1}} \longrightarrow R_{h^{-1} \odot g^{-1}} E_g .$$

The other Yang–Mills type action results from taking the functional θ_2 (29). The action reads:

$$S_{\rm YM} = \int\limits_G \sum_{g,h} E_g E_{h^{-1}} F^\star_{g(g^{-1})} F_{h(h^{-1})} \,. \tag{34}$$

In this case if we take $\theta_1(F, F^\star)$ we shall obtain the same action.

Finally we are able to construct the action linear in F,

$$S_m = \left(\int \mathrm{Tr} \sum_g E_g F_{g(g^{-1})} \right) . \tag{35}$$

Let us point out that the constructed actions or rather each possible linear combination of them may pretend to be the effective action of our theory. This ambiguity appears only due to the noncommutativity of the differential algebra. Another significant feature of the theory is that the space of possible metrics on Ω^* is much smaller than one would expect.

4. Examples

In this section we shall briefly discuss two simple examples of the unitary gauge theory on the two- and three-point spaces. We shall construct the action functionals and discuss briefly the solutions and their geometry.

4.1. Gauge theory on $\mathbb{Z}_2$

Let us denote the group elements of $\mathbb{Z}_2$ by $+$ and $-$. We take the group H to be $U(N)$ and the algebra A to be the algebra of complex matrices M_n. Since the structure group is unitary, the connection must be antihermitian, therefore we obtain the following relation,

$$\Phi_-(+) = \Phi^\dagger_-(-), \tag{36}$$

and the same applies to $\Psi_- = 1 + \Phi_-$. Thus, effectively we have got only one degree of freedom, which is an arbitrary complex matrix. We take it as $\hat{\Psi} = \Psi_-(+)$ and for convenience we drop here the subscript.

Let us observe that for $n \geq 1$ all Ω^n are one-dimensional. Consequently, the curvature two-form $F = \chi^- \otimes \chi^- F_{--}$ is completely determined by one coefficient function F_{--}, which using Eqs. (23), (36) can be calculated,

$$F_{--}(+) = -(\hat{\Psi}^\dagger \hat{\Psi} - 1), \tag{37}$$

$$F_{--}(-) = -(\hat{\Psi} \hat{\Psi}^\dagger - 1). \tag{38}$$

The metric is set by one function on $\mathbb{Z}_2$, E_-; for simplicity we assume that it is constant and equal to 1. Now one can easily see that all possibilities for the Yang–Mills action are reduced to the following,

$$S_{\mathrm{YM}} = \mathrm{Tr}(\hat{\Psi}^\dagger \hat{\Psi} - 1)^2, \tag{39}$$

where we have already done the Haar integration.

This has exactly the form of the potential of the Higgs model and was first obtained in Connes' consideration of the $\mathbb{C}^2$ algebra [2]. Here, however, we can modify this action slightly by adding the term linear in F (35), proportional to $\mathrm{Tr}(\hat{\Psi}\hat{\Psi}^\dagger - 1)$. The physical meaning of this term is very important, without it we should get a relation between the Higgs mass and other parameters. However, if this term is present, the Higgs mass is still a free parameter of the model.

Let us now make short comments on the moduli space of the theory and the extremal points of the action functionals. The space of flat connections modulo gauge transformations is trivial. Indeed, the vanishing of F is equivalent to the unitarity of $\hat{\Psi}$ and from its transformation rule (21) we see that arbitrary $\hat{\Psi}$ can be obtained from the trivial flat connection, $\Psi = 1$, by choosing the appropriate gauge transformation. The Yang–Mills action has one absolute minimum, which is reached for the flat connections.

4.2. Gauge theory on $\mathbb{Z}_3$

Let us denote the elements of the group by $0, 1, -1$ and the group action by $+$. The space of one-forms is two-dimensional, spanned by the basis of invariant forms χ^+, χ^-, (for the indices, $+$ stands for $+1$ and $-$ for -1).

We choose the metric η to be in its simplest form, so that E_+ and E^- are both constant. The algebra of derivations ∂_+, ∂_- obeys the following relations,

$$\partial_-\partial_- = 2\partial_- - \partial_+, \tag{40}$$

$$\partial_+\partial_+ = 2\partial_+ - \partial_-, \tag{41}$$

$$\partial_-\partial_+ = \partial_+\partial_- = \partial_+ + \partial_-, \tag{42}$$

which determine the structure constants and the rules of the differential calculus in this case (11).

Now, let us again construct the $U(N)$ gauge theory. The gauge potential one-form Φ could be expressed as $\chi^+\Phi_+ + \chi^-\Phi_-$. The condition that Φ is antihermitian (20) takes the form,

$$\Phi_+(g) = \Phi_-^\dagger(g+1), \quad g \in \mathbb{Z}_3. \tag{43}$$

From this relation we see that the connection is completely determined by either of its coefficients. Let us define $\Psi = 1 + \Phi_+$, and use it in our further analysis. Its gauge transformation is as follows,

$$\Psi(g) \rightarrow H^\dagger(g+1)\Psi(g)H(g), \quad H(g) \in U(N), \; g \in \mathbb{Z}_3. \tag{44}$$

Now we can express the curvature in terms of the function Ψ.

$$F_{++} = (R_-\Psi^\dagger) - (R_+\Psi)\Psi, \tag{45}$$

$$F_{+-} = 1 - (R_-\Psi)(R_-\Psi)^\dagger, \tag{46}$$

$$F_{-+} = 1 - \Psi^\dagger\Psi, \tag{47}$$

$$F_{--} = \Psi - (R_+\Psi^\dagger)(R_-\Psi^\dagger) - \Psi. \tag{48}$$

One can easily notice that $F_{--} = R_+F_{++}^\dagger$ and both F_{+-} and F_{-+} are hermitian.

Before we discuss the action functionals let us find the moduli space of flat connections in this case. If F vanishes, from (46) we obtain that the function Ψ must be unitary, whereas $F_{++} = 0$ gives us from (45) and from the previous result the following identity,

$$(R_+\Psi)\Psi(R_-\Psi) = 1\,. \tag{49}$$

Using the transformation rule (44) and the condition above we can again show that all flat connections are gauge equivalent.

Finally, let us present the actions. The action linear in F is,

$$S_m = 2\int \mathrm{Tr}(E_+ + E_-)\left(\Psi\Psi^\dagger - 1\right), \tag{50}$$

and we are left with two possibilities for the Yang-Mills type quartic action,

$$S_1 = \int \mathrm{Tr}\left((2E_+E_-(1-\Psi^\dagger\Psi)^2) + (E_-^2 + E_+^2)(1_-\Psi^\dagger\Psi)R_-(1-\Psi^\dagger\Psi)\right), \tag{51}$$

$$S_2 = \int \mathrm{Tr}\left(2E_+E_-(1-\Psi\Psi^\dagger)^2 + (E_-^2+E_+^2)\left(\Psi\Psi^\dagger(1+R_+(\Psi\Psi^\dagger))\right.\right.$$
$$\left.\left. - R_-(\Psi)\Psi R_+(\Psi) - R_+(\Psi^\dagger)\Psi^\dagger R_-(\Psi^\dagger)\right)\right) . \tag{52}$$

Each linear combination of them may pretend to be the global action of the theory. Let us observe the remarkable property that the second action (52) contains a third order polynomial in Ψ, and a certain combination of S_1, S_2 and S_m may be in fact such a term:

$$S_3 = S_1 - S_2 - cS_m = \int \mathrm{Tr}(E_-^2 + E_+^2)\left(R_-(\Psi)\Psi R_+(\Psi) + \text{c.c.}\right), \tag{53}$$

where c is a constant $c = (E_+^2 + E_-^2)/(E_+ + E_-)$.

This property is rather surprising, and though it does not appear in the $\mathbb{Z}_2$ case, we may expect that it is common for discrete differential calculus.

The problem of the extremal points of the presented actions is more complicated than in the previous example and in some cases the action might not even have an absolute minimum.

5. Symmetries and subalgebras

This section is devoted to a brief discussion of possible restrictions of the theory, which arise from considering subalgebras of the constructed differential algebra.

Suppose we take a proper subalgebra of $\tilde{\mathcal{A}}$, we shall denote it by $\tilde{\mathcal{B}} \in \tilde{\mathcal{A}}$. Then we can find a graded differential subalgebra of $\tilde{\Omega}(\mathcal{A})$ in such a way that the zero-forms are the elements of $\tilde{\mathcal{B}}$ and all the rules of differential calculus remain unchanged. For instance, let us consider the subalgebra of $\mathcal{A}$ which contains all $\mathbb{C}$-valued constant functions on the group G and denote it by $\mathcal{A}_0$. If we take as Ω^n all differential forms that have their coefficients in $\mathcal{A}_0$, we obtain the required subalgebra of Ω^*. Of course, Ω^1 is no longer generated by the image of d. We can express that construction in a more formal way. Indeed, if we use the group of automorphisms of $\mathcal{A}$, $R_g, g \in G$, which can be easily extended to the whole of Ω^*, we see that Ω_0^* remains invariant under the action of this group, so the obtained differential algebra is the one of invariant differential forms. Let us observe that although Ω^* did not depend on the group structure of the discrete base space, the invariant differential algebra Ω_0^* depends on it, as for different groups we should have different groups of automorphisms R_g.

A trivial example, the differential calculus on Lie groups, is set by the restriction to the algebra of left-invariant forms. By taking this invariant differential algebra as our starting point we may develop a gauge theory similarly as we have proceeded with our earlier examples. In particular, if one considers a $SU(2)$ gauge theory on the invariant differential algebra of the $SU(2)$ group one recovers a model of a gauge theory on a matrix algebra with inner derivations [7-9]. It suggests that such a model is in fact a kind of Kaluza-Klein theory, with additional restriction that the additional dimensions are in a group manifold and the physics is invariant with respect to its automorphisms.

Let us turn our attention back to the examples of discrete geometry. Having the differential grading algebra Ω_0^* we can proceed with the construction of gauge theories invariant with respect to the action of the discrete group. The gauge transformations are now global, i.e. they are also constant when considered as functions on G. The same applies to the coefficients of the connection and the curvature. Let us see what the effect of this is for the constructed theories on $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

In the first case, from (36) we get that Ψ must be hermitian. This implies that the minima of the Yang–Mills action (39), which again correspond to the flat connections, are separated. The moduli space is equal to the space of equivalence classes of unitary and hermitian matrices. Notice that for $U(1)$ we obtain $\mathbb{Z}_2$, which is the base space of our theory G. The same applies to the model on $\mathbb{Z}_3$. This time, the moduli space of flat connections is the space of equivalence classes of matrices that satisfy the relation $\Psi^3 = 1$. Again, for $U(1)$ this space can be identified with $\mathbb{Z}_3$.

Let us point out that by restricting ourselves to the invariant differential algebra on $\mathbb{Z}_3$ we obtain an effective gauge theory with one field and a Higgs type potential. The difference between this and the $\mathbb{Z}_2$ case lies in the form of the potential function and we can see that the $\mathbb{Z}_3$ case does not correspond to the effective potential of the Standard Model.

We shall end here the discussion of possibilities arising from employing the symmetry principles encoded in the automorphisms of the algebras. Our aim was only to show this option and briefly discuss its implications for the given models.

6. Conclusions

We presented in this paper a systematic approach to the problem of constructing gauge theories on discrete spaces. This involved the introduction of the differential calculus, which we have carried out for spaces equipped with the structure of a finite group. Let us point out that the group structure becomes important only when considering some invariant quotients of the differential algebra.

We introduced the notion of the metric in noncommutative geometry using only the differential calculus. This might be the starting point for considering gravity on discrete spaces; also the connection to the approach based on the Dirac operator would be interesting.

We presented the construction of gauge theories only for unitary gauge groups; however the formalism could be easily extended to arbitrary groups. In fact, they do not have to be continuous and one may as well use discrete groups for this purpose. Another spectacular property of this theory is the fact that we can take as a starting point the algebra, which is not necessarily the algebra of $\mathbb{C}$-valued functions or functions valued in any algebra but its proper subalgebra. For instance, if have an algebra A, and its subalgebras, say $A_0, A_1, \ldots$, we can construct the proper subalgebra of $\bar{\mathcal{A}}$ as the set of functions such that $f(x), x \in G$, takes values in A_j for some index j. Now, following the same steps as we presented in this paper, we can construct the differential calculus and the gauge theories. It appears that if we consider the two-point space, as in the first example, with the algebra of functions taking values in $\mathbb{C}$ at one point and in $\mathbb{H}$ at the other, which is the algebra of quaternions, its product with the continuous geometry of the Minkowski space gives us the precise description of the pure gauge part of the electroweak interactions. Of course, in this approach fermions stay out of the picture.

The discussion of the metric has led us to the actions of the considered models. This seems to be another interesting point for future investigations since the existence of more gauge invariant quantities is a remarkable property of this theory.

Finally, let us notice that the restriction to finite groups may be relaxed as well and one can analyse similar models for infinite discrete groups like $\mathbb{Z}_n$, for example.

The program of noncommutative geometry has given us the possibility of considering a far more general class of models than the one arising from the analysis on manifolds. Since quantum groups and discrete geometry are the two most interesting and promising examples, their study seems to be important and we believe that their analysis, in particular the investigation of gauge theories in this framework, will help to gain a better understanding of the subject.

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